CWI Quarterly Volume 8 (3) 1995, pp. 237 – 255

The Extended Parameter Space of a Model for 1:4 Resonance

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The $\mathbb{Z}_4$-equivariant planar vector field $\dot{z} = e^{i\alpha}z + e^{i\varphi}z|z|^2 + b\bar{z}^3$ models the dynamics near a closed orbit losing its stability in 1:4 resonance. It is known that there are at least 48 regions in the (b, φ)-plane of constants, corresponding to equivalence classes of unfoldings in the parameter α. It is a conjecture by Arnol'd that there are not any more such regions.

We propose to disregard the distinction, typical for models from bifurcation theory and many applications, between unfolding parameters and constants determining the nonlinear terms. In this spirit we present the bifurcation set for the above model in (b, φ, α)-space, which represents all known information in a condensed way. This approach leads to new support for the above conjecture, notably through the study of bifurcations at infinity of the phase space and the use of numerical techniques.

1. The periodically forced oscillator

This paper is derived from my Ph.D.-thesis [18], which contains a more complete treatment of the problem discussed here. Further details can be found there and in [16], [17], [19]. As general references for the concepts of dynamical systems and bifurcation theory see [2], [3], [11], [13].

Consider the up-and-down motion of the front wheel of a motor bike riding on an evenly corrugated road. The wheel is suspended by a damped nonlinear spring and is excited as the bike rides along. (All effects of other parts of the bike are neglected.) There are two parameters in the system: the amplitude and the frequency of the excitation, given by the 'badness' of the road and the speed of the motor bike, respectively.

If the road is not too bad the wheel follows its shape and we observe a periodic motion of the wheel in the external frequency of the corrugation. As the road gets worse, typically an additional internal oscillation with small amplitude appears on top of the original one: the wheel starts bouncing a little. The original periodic motion is now unstable, and is consequently not observed. For certain speeds the two frequencies are in resonance, meaning that their ratio is a rational number p/q. Important resonances occur for $q = 1, 2, 3, 4$. Close to such a resonance the wheel may exhibit more complicated motions, which can have catastrophic effects on both rider and bike. We are concerned with the most complicated and the only unsolved case of *1:4 resonance*, when the external frequency is exactly a quarter of the internal frequency as the originally stable periodic motion disappears.

The front wheel is an admittedly not perfect example of a *periodically forced oscillator*. Concrete examples are the forced Van der Pol equation and the damped forced pendulum. By setting $z = x + i\,\dot{x} \in \mathbb{C}$ and assuming that the forcing has the constant frequency one, the forced oscillator can be written as the vector field

$$\begin{aligned} \dot{z} &= f_{\mu,\beta}(z, \bar{z}, t) \\ \dot{t} &= 1 \end{aligned} \tag{1}$$

on the phase space $\mathbb{C} \times (\mathbb{R}/\mathbb{Z})$.

By performing a suitable change of coordinates, we may assume that $\{z = 0\}$ is a periodic orbit, independently of the parameters μ and β. The question is what happens as this orbit loses its stability, which can be studied by means of the *Poincaré map* $P_{\mu,\beta}$. In the present situation this is simply the stroboscopic map of the forcing period (which was set to one), that is, $(P_{\mu,\beta}(z), 1) = \phi^1_{\mu,\beta}(z, 0)$, where $\phi^t_{\mu,\beta}$ is the flow of (1). We may assume that $DP_{\mu,\beta}(0)$ has eigenvalues $e^{\mu \pm 2\pi i \beta}$, so that the loss of stability of the origin occurs for $\mu = 0$. Under the genericity condition $\beta \neq p/q$ for $q = 1, 2, 3, 4$ an invariant circle bifurcates from the origin for $\mu = 0$ in a Hopf bifurcation for maps, also called a Neimark-Sacker bifurcation. The question is what happens if the genericity condition $\beta \neq p/q$ for $q = 1, 2, 3, 4$ is violated, in which case one speaks of a *strong resonance*.

2. A MODEL SYSTEM

If $\mu = 0$ and $\beta = p/q$, where p and q are relatively prime, system (1) undergoes a Hopf bifurcation with *p:q resonance* and the linear part $DP_{\mu,p/q}(0)$ is the rotation over the angle $2\pi\,p/q$. The key tool in this situation is a classical normal form theorem that reduces the problem to the study of a *planar vector field* by averaging away the time-variable t in (1); compare [8].

THEOREM 1 ([1]; [22])
In a neighborhood of $(z, \mu, \beta) = (0, 0, p/q)$ the map $P_{\mu,\beta}$ can be approximated up to any prescribed order by the time-one map of a $\mathbb{Z}_q$-equivariant planar vector field $X_{\mu,\beta}$, composed with $J_{p/q} := DP_{0,p/q}(0)$, the rotation over $2\pi p/q$.

This theorem reduces the p:q resonance problem to finding all *versal unfoldings* of codimension two of $\mathbb{Z}_q$-equivariant vector fields. An unfolding describes all possible dynamics near the singularity in question with the amount of parameters given by the codimension. It is called versal if it is the most general unfolding in some sense; for the rather technical definitions see [1], [18]. Except for $q = 4$, these versal unfoldings are known; relevant references are [1], [2], [3], [6], [7], [9], [15], [22]. As was mentioned earlier, for $q \geq 5$ an invariant circle is born in the Hopf bifurcation just like in the absence of resonance; one speaks of *weak resonances* in this case. The case of 1:4 resonance marks the transition between the weak and the strong resonances. For $\mathbb{Z}_4$-equivariant planar vector fields there is a well-known conjecture.

CONJECTURE 2. *[1]*
All versal unfoldings of codimension two of a $\mathbb{Z}_4$-equivariant planar vector field are contained in the model equation

$$\dot{z} = \varepsilon z + Az|z|^2 + B\bar{z}^3, \tag{2}$$

where $\varepsilon, A, B \in \mathbb{C}$.

As part of this conjecture Arnol'd found 48 regions in (A, B)-space of unfoldings in the parameter ε. We will see later that this gives a total of eleven different unfoldings if one takes into account some additional symmetry of phase portraits. It has not been proved that there cannot be other than the known unfoldings and that they are versal. It is the purpose of the approach presented here to give arguments in favour of Conjecture 2. Note that the two nonlinear terms in (2) are of the same order, so that their relative influence is determined exclusively by the coefficients A and B. This is the reason why this case is the most difficult one.

System (2) has a four-dimensional (A, B)-space of constants and a two-dimensional ε-plane of the unfolding parameter. By scaling the phase plane one can see that two real constants are enough to determine the nonlinearity. Furthermore, all bifurcation curves in the ε-plane are straight lines from the origin, so that considering only what happens for values on the unit circle $\varepsilon = e^{i\alpha}$ in the ε-plane still gives all information about an unfolding. We choose to work with the reduced equation

$$\dot{z} = e^{i\alpha} z + e^{i\varphi} z\,|z|^2 + b\bar{z}^3, \tag{3}$$

where $b \in \mathbb{R}^+$ and $\alpha \in (-\pi, \pi]$. Due to reflectional symmetries in phase space, it is sufficient to consider the case $\varphi \in [\pi, 3\pi/2]$. In the literature the reader will find the equivalent system $\dot{z} = e^{i\alpha} z + Az\,|z|^2 + \bar{z}^3$. We use (3) because the interesting behavior occurs in a compact piece of parameter space; see [17], [18] for a geometrical interpretation of the two reductions.

The *bifurcation sequence* for fixed (b, φ) is the sequence of topologically different phase portraits as α varies. Two bifurcation sequences are *equivalent* if the same types of bifurcations occur in the same order and the respective phase

FIGURE 1. The (b, φ)-plane of (3) with the equivalence classes of different bifurcation sequences (roman numerals).

portraits are topologically equivalent. Clearly, two unfoldings are equivalent if the corresponding bifurcation sequences are. The problem can now be stated as follows.

1. Find *all* equivalence classes of bifurcation sequences in the (b, φ)-plane.
2. Show that they represent versal unfoldings.

This was studied in [2], [5], [21] (for the equivalent model), which lead to the picture of the (b, φ)-plane as illustrated in Figure 1. The list of all known bifurcation sequence can be found in [16], [18].

3. THE EXTENDED PARAMETER SPACE

Because we believe that our approach can be useful in other situations, we present it in a general setting. Consider a polynomial model system of *fixed degree*

$$\dot{x} = F(x; \lambda, c), \tag{4}$$

where $x \in \mathbb{R}^n$ is a phase variable, $\lambda \in \mathbb{R}^m$ is an unfolding parameter and $c \in \mathbb{R}^l$ is a constant. One can think of F as a polynomial model or as the truncation of a Taylor series in normal form. The general question is: what is the behavior of (4) as λ is varied, once c has been fixed to a non-exceptional, or generic value? In other words, one wants to know the partition of c-space into regions of equivalent unfoldings in λ. The values where c is exceptional are of particular importance as they form the boundaries between different regions in c-space.

The clear distinction between the unfolding parameter λ and the constant c is quite typical. We propose to give up this distinction and to study the bifurcation set in the extended parameter (c, λ)-space. The bifurcation set divides this space into regions of topologically equivalent phase portraits, an easier notion than the equivalence of unfoldings. By projection in the direction of λ the information on the equivalence classes of unfoldings in c-space can be retrieved from the bifurcation set. By "drilling" in the direction of λ for a fixed c the unfolding can be found.

Also in the context of applying numerical techniques it is very natural to consider the extended parameter space. Drawing phase portraits numerically allows one to get an idea of how this space is structured. Furthermore, the important idea of following an object in phase space by continuation under the variation of parameters typically gives a new object in the product of the phase space with the extended parameter space; see Section 6 for more on numerical methods.

The boundaries in c-space are either projections of bifurcations of codimension $(m+1)$ or correspond to bifurcations at infinity of the phase space. The fact that bifurcations at infinity become important may be somewhat surprising. However, typically there are boundaries in c-space, crossing which leads to the escape of equilibria or limit cycles to infinity in phase space. This can be described by bifurcations at infinity, that is, by bifurcations at the boundary of the phase space $\mathbb{R}^n$ of (4). The study of bifurcations at infinity can be very useful for finding all phase portraits. (Recall that (4) is of fixed degree. To find all phase portraits we do not need to add higher order terms.) We note here that often only those bifurcations at infinity are of importance that lead to a topological change of phase portraits in the phase space $\mathbb{R}^n$.

The general program for finding the bifurcation set of (4) is the following. First one calculates all *local* hypersurfaces for which one knows parametrizations. These may include bifurcations at infinity. Then one uses topological arguments and numerical continuation to find the *nonlocal* hypersurfaces for which no parametrizations are known, such as for example those corresponding to saddle connections. The bifurcation set represents all information in a condensed form. Unfortunately, visualizing (c, λ)-space has its limits if the dimensions m and l are too big.

Surface	Characterizing property
$\odot_1$ $\odot_2$	First and second Hopf bifurcation at 0
S_1 S_2	First and second saddle-node bifurcation
T_+	Hopf bifurcation at secondary equilibria
$\pitchfork^\infty$	Pitchfork bifurcation at ∞
S^∞	Saddle-node bifurcation at ∞
$\odot^\infty$	Hopf bifurcation at ∞
♀	Homoclinic loop at secondary equilibria
$\square_1$ $\square_2$	First and second square connection
⌘	Clover connection
$\circledcirc$	Saddle-node of limit cycles

Curve	Characterizing property
$\odot\mathrm{S}$	Hopf bifurcation at 0 coincides with second saddle-node bifurcation
BT	Bogdanov-Takens bifurcation
⌘T	Clover connection with zero trace
⌘S_1	Clover connection coincides with first saddle-node bifurcation
$\square\mathrm{S}_1$ $\square\mathrm{S}_2$	Square connection coincides with first or second saddle-node bifurcation

TABLE 1. Symbols for surfaces of codimension-one bifurcations and for curves of codimension-two bifurcations.

4. THE BIFURCATION SET

We now apply the ideas from the last section to the model for 1:4 resonance (2). The dimensions are ideal since we are dealing with a three-dimensional extended (b, φ, α)-space. Furthermore, the phase space is two-dimensional which largely facilitates the study of bifurcations at infinity. We present all known surfaces of codimension-one bifurcations in the bifurcation set, which divide (b, φ, α)-space into regions of topologically equivalent phase portraits. For details and proofs we refer to [17], [18]. The symbols we use to label the surfaces can be found in Table 1. We begin by presenting the local surfaces in the bifurcation set.

LEMMA 3. *The following local surfaces are in the bifurcation set.*

(a) Two planes $\odot_1$ and $\odot_2$ of Hopf bifurcations at 0 given by $\alpha = \pm\pi/2$.

(b) Two surfaces S_1 and S_2 of saddle-node bifurcations given by $\alpha = \varphi - \pi \mp \arcsin b$, where $0 < b < 1$.

(c) The surface T_+ of Hopf bifurcations of secondary equilibria, where the trace is zero at the nodes, given by

$$\tan\alpha = \frac{\sin\varphi - \sqrt{b^2 - \cos^2\varphi}}{2\cos\varphi}, \text{ where } \pi + \arccos\sqrt{\frac{b^2(1-b^2)}{3b^2+1}} < \varphi < \frac{3\pi}{2}.$$

FIGURE 2. Possible saddle connections of codimension one, called a *square connection* (left) and a *clover connection* (right), respectively.

(d) *The plane* $\pitchfork^\infty$ *of pitchfork bifurcations at* ∞ *given by* $b = 1$.

(e) *The surface* $\odot^\infty$ *of Hopf bifurcations at* ∞ *given by* $\varphi = 3\pi/2$, *where* $b \in [0, 1)$.

The local surfaces form the skeleton of the bifurcation set. They intersect each other and also the yet unknown nonlocal surfaces in lifts of the boundary curves in the (b, φ)-plane of Figure 1. The nonlocal surfaces and their intersection curves with other surfaces involve saddle connections of square and clover type as depicted in Figure 2.

LEMMA 4. *On the local surfaces one finds the following curves of codimension-two bifurcations. (We use the same notation for the lifted curves as for their projections.)*

(a) *The curve* $\odot\mathrm{S}$, *given by* $\varphi = \pi + \arccos b$, $\alpha = \pi/2$, *where* S_2 *and* $\odot_2$ *intersect.*

(b) *The curve* BT *of Bogdanov–Takens bifurcations, given by* $\varphi = \pi + \arccos\sqrt{\frac{b^2(1-b^2)}{3b^2+1}}, \alpha = \varphi - \pi - \arcsin b$, *where* S_1 *meets* T_+.

(c) *The system is Hamiltonian along the intersections of* $\odot_1$ *and of* $\odot_2$ *with the plane* $\{\varphi = 3\pi/2\}$, *given by* $\varphi = 3\pi/2, \alpha = \pm\pi/2$.

(d) *The curves* $\square\mathrm{S}_1$ *on* S_1 *and* $\square\mathrm{S}_2$ *on* S_2, *where there is a square connection at the moment of the respective saddle-node bifurcation.*

(e) *The curve* ⌘S_1 *on* S_1, *where there is a clover connection at the moment of the first saddle-node bifurcation.*

(f) *The curve* ⌘T *lies on a surface where the trace is zero at the saddles, which is not part of the bifurcation set, but can be parametrized like* T_+.

All nonlocal surfaces were shown to bifurcate from Hamiltonian lines in [21]. This information was important for finding all unfoldings. We use this together with topological arguments to find the global structure of the nonlocal surfaces, where we assume that the known surfaces intersect only in the known curves of codimension-two bifurcations. This assumption can be checked by numerical techniques; see Section 6.

THEOREM 5. *The following nonlocal surfaces are in the bifurcation set.*

(a) *Two surfaces* $\square_1$ *and* $\square_2$ *of square connections. The upper surface* $\square_2$ *extends from the curve* $\square\mathrm{S}_2$ *on* S_2 *to the Hamiltonian line* $\varphi = 3\pi/2$, $\alpha = \pi/2$, $b \in [0, 1]$. *The lower surface* $\square_1$ *extends from the Hamiltonian line* $\varphi = \pi/2$, $\alpha = -\pi/2$ *to the curve* $\square\mathrm{S}_1$ *on* S_1 *for* $b \leq 1$, *and to the Hamiltonian line* $\varphi = 3\pi/2$, $\alpha = -\pi/2$ *for* $b > 1$.

(b) *The surface* ☘ *of clover connections, extending from the curve*